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Cuisenaire rods are colored, rectangular pieces of wood one unit wide and any number of units long. The color of the rod depends upon its length. Rods can be used in kindergarten through high school, but are particularily helpful in the early grades. The rods can be used to teach children such concepts as "plus," "minus," "equals," multiplication, and factors. Ways of using the rods to teach the basic mathematical concepts are described.

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An Introduction to the

CUISENAIRE RODS

An Approach to the Teaching of Elementary Mathematics

By

WILLIAM P. HULL

SECOND PRINTING, MAY 1962

NATIONAL COUNCIL OF INDEPENDENT SCHOOLS

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An Introduction to the . . .

CUISENAIRE RODS

By William P. Hull

Shady Hill School

The colored rods invented by Georges Cuisenaire of Thuin, Belgium, have become one of the teacher's most valuable tools. There are a number of uses for these rods from kindergarten through high school, but they are particularly helpful in the early school years, where they make possible a rather different curriculum than we have previously thought possible for young children. A more flexible study of the numbers from one to ten, for example, leads easily to a consideration of multiplication and the fractional parts of these numbers. By helping a child develop an appreciation for general number relationships, still within this restricted range of the first ten numbers, we can simplify his task of learning the specific combinations and prepare for more rapid future progress. It is a bit startling to find first graders solving such equations as $7 = 3 + \frac{1}{3} \times \square + 2$, involving unknowns which are not at the end of the equations, multiplication by fractions, and order of operations. When one realizes that the numbers here do not exceed ten, and that this is merely a way of writing what the child can easily see from the rods, these activities are not so surprising. What has been achieved is a more flexible way of looking at the same subject material which is traditionally taught in the early grades. While the program which becomes possible may sound ambitious in relation to the expectations which we usually have had for children in these grades, we are finding that these things can be accomplished normally, without a great deal of strain, and that the student's growing awareness of the relatedness of ideas and his enthusiasm for mathematics provide a good basis for future learning.

It should be pointed out that the Cuisenaire materials do not, by themselves, constitute a system of teaching mathematics which must be followed sequentially, step by step, in order to be effective. They may be used at many different levels, from kindergarten through high school, with entire classes of typical children or with special groups of either gifted or retarded children. While the rods make it possible to go more deeply into mathematics at an earlier level than was previously possible, children at the same time can be learning the traditional subject matter of the early grades in ways which will be of great value to them later on. It would be unfortunate if teachers felt that if rods were to be used at

all they must be used systematically through a school system. Since there can be many beginnings, with the children starting with what they see in front of them and going on from there, the teaching of mathematics need not follow a strict linear sequence. Certain aspects of geometry, algebra, the theory of sets, exponents, etc., can be more profitably approached through the rods than in any other way at a variety of levels, with the rods being used for one special topic and no other. While we will be describing different ways in which they can be used to provide the basis for class work in the early grades, we do not wish to imply that there is but one sequence which must be followed step by step. Furthermore, if the rods are used as the basis for class work, it does not mean that the children will have to use them all the time. The students are usually the best judges of when they are needed, and will be ready to give them up when they find it easier to do the operations which are called for in their heads, returning to the rods when a new type of problem is encountered or when there is a need to check on something which they are not sure about. In many cases the teacher will find that children can learn the same things better with the rods than without them, but when better ways are found we will certainly want to use them. In giving some idea of the progression which is possible we recognize that the best results will be obtained when the teacher is free to respond to the interests and needs of her pupils, and to teach in a way which makes use of her own sources of vitality, rather than one which is imposed by over-rigorous curriculum specifications.

Dr. Caleb Gattegno, former Professor of Pure and Applied Mathematics at the University of London, has devoted much of his time to developing the use of the Cuisenaire rods. His series of books for children, *Arithmetic with Numbers in Color*, consists of questions which children can answer by using the rods. The first book may be used with first grade children or even younger, though some of the problems will have to be read to them at this stage. Older children can frequently profit from much of the same material contained in Book I; considerable flexibility is called for despite the limitation on content. The sequence of material in these books is cleverly arranged, and, though some of the material may appear formidable to the casual observer, it is well within the range of children who have had some experience with the rods. Since Dr. Gattegno's books contain many detailed suggestions for activities, this paper is intended only as a general introduction to the rods with some observations about how they can be used most effectively.

The physical properties of the rods make possible a series of operations which are actually carried out by the child. At first his understanding will be at the perceptual level without having to be directly dependent upon words, the physical action and perception providing a model for the mental operation which will take its place. While we will attempt to indicate how the development of mathematical thought can proceed through these actions and perceptions, it is necessary to point out that

there is another aspect which is most important. Young children learn to meet the demands which are made upon them by adults in a variety of ways. Frequently their desire to please and their confusion about what is expected of them lead them to abandon their own common sense. It is as though there were two sets of rules, one for behavior in school and the other for behavior in other situations. When the teacher is the final authority in intellectual as well as behavioral matters the child's reasoning is often directed to meeting the demands of the situation instead of to considering the intrinsic nature of the problem. Under these conditions progress is very slow, with many misunderstandings, and much repetition is needed. When, on the other hand, children are given more intellectual freedom, so they do not have one set of rules for their behavior in school and another set for their own play which is vital to them, their progress will be of quite a different nature. When the conditions of the classroom encourage intellectual playfulness, children of six and even younger can experience the thrill and excitement which goes with discovery and some of the satisfactions which a creative mathematician enjoys. A substantial factor in the success of children who use these materials is that they can make a rather different approach to learning than is usually encouraged in the early grades.

Another factor which strongly influences the kind of progress which can be made concerns the use of language. Although we are never called upon to explain many of the complex things we have learned to do, we are frequently forced to evaluate a child's understanding by the rearrangement of words which he makes. In situations where a child feels considerable pressure for achievement he may be tempted to memorize a procedure and an explanation to go with it. While children who are gifted with some facility can use such an approach effectively as long as the same patterns are repeated, they may have little real grasp of the material and be lost when they are asked to apply what they have presumably learned to another situation. In mathematics there are often many different ways of looking at the same problem. When it is a case of actually *seeing*, alternatives are not destroyed by focusing on one particular relationship, but when we approach the same problem through words we are forced to deal with ideas one at a time, and the other relationships which exist are apt to be lost. This is the reason that many problems can be grasped at the perceptual level before they can be expressed in words. Since it is possible to act upon an awareness before being able to verbalize it, we should not be surprised that children can become quite competent in certain activities before they can explain what they are doing. Many of the confusions which children have are essentially confusions with words. Work with deaf children has shown that, with the help of the rods, a great deal of mathematics can be learned without any words at all. Normal children can progress more rapidly and handle more complexity when the things they talk about are based on the richness of their own activities. Since what a child learns with

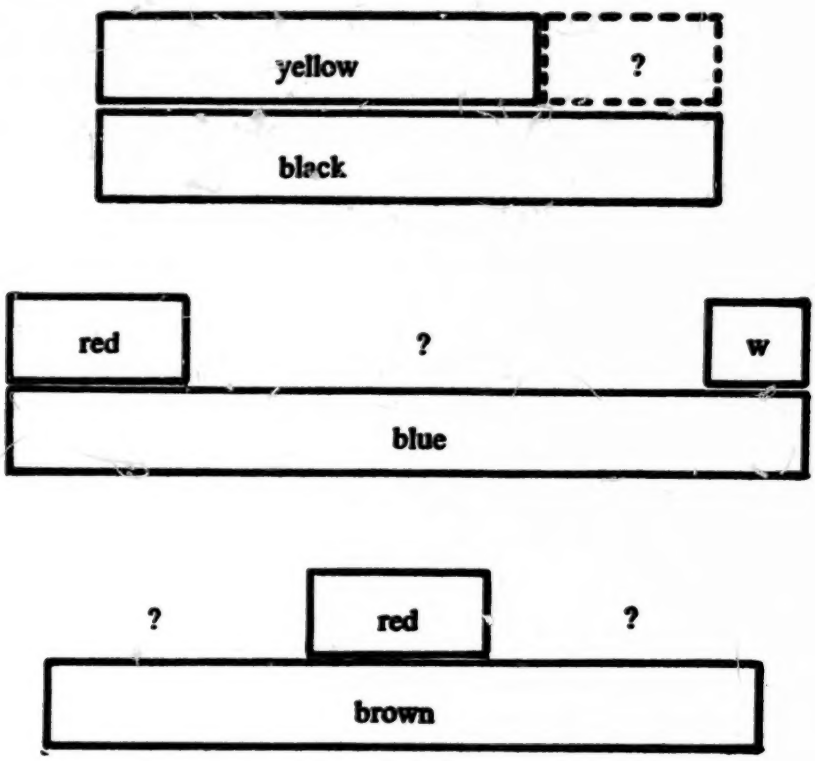
the rods is based upon his own experience, he can always re-construct what he has done and is not dependent upon the memory of a particular sequence of words for his success.

The rods invite exploration. The urge to play with them, to fit them together in various ways, to make designs or representational structures is usually strong. A wide variety of activity will be observed in any given class. Some children will make various constructions with the rods flat on the desks, while others enjoy experimenting with asymmetrical structures; some children are ready to invent new arrangements every time they use the rods, while others can be observed repeating the same patterns again and again for a period of time. Tours can be arranged so that the students have a chance to look at each other's constructions. When discoveries made by individuals are shared by the whole class interest and involvement will be high.

Many different arrangements for this free play are possible. Each child may be given a sack of rods, or a number of children can work together with a larger set, at a table or with the desks pushed together. If the children are going to work by themselves they will need a minimum of one fifth of a set. If they are divided this way, however, it is desirable to have several larger sets available for the projects which require more rods. The time spent on free play may vary considerably from class to class, or from day to day within the same class. If a half hour period is used for mathematics in the first grade, for example, the teacher may wish to spend the first fifteen minutes of each period for several weeks on free play before beginning more formal classroom work.

The color of the rods gives us an easy way to identifying their length, but children can soon learn to judge the length by feel or by sight. If they are asked to take the white, the red, the light green, and the crimson and hold them out of sight in their laps, they will be able to hold up any of the rods as they are called for by color. To judge size in this way the rod should be held in the closed hand so that it crosses the palm diagonally. At first the judgment will be made in relation to the other rods which are being held, but before long many children will know just by the feel of the rod which one it is. Blind children, operating by sense of touch alone, can develop very clear ideas about number. We do not know how useful it is for children who can see to learn to make some of the same discriminations, but when such learning goes rapidly it is probably worth the time spent. An interesting problem is to name the color of a rod before it is drawn from a bag. If a set of rods are arranged in a stair in front of a child while he is doing this he will have an opportunity to co-ordinate his judgment of sight and feeling. At first, though, he may need to feel one or more of the rods in the stair with his other hand before he is sure of the one he is holding in the sack. Aside from the practical advantage of being able to estimate the length of small objects by sight or touch, such experiences can contribute much to the vitality of a child's experience with number.

In their free play children will be learning to make judgments of length by eye. This skill becomes very useful when they work out the pattern of each rod because it enables them to think of number relationships and to solve problems without having to resort to counting. Children who are taught to solve problems by counting are often reluctant to give this method up in favor of more efficient ways of operating. Though they spend considerable time memorizing combinations, they are frequently not able to use what they have learned in other situations, and find it difficult to think about problems which they have not been taught specifically how to solve. With the rods the first problems are solved by sight. The child knows immediately whether his solution is correct or not without having to count, and furthermore he can *see* that certain relationships will help him figure out what he needs to know if he should forget. The game of placing two rods side by side and finding what other rod is needed to make the same length is very helpful in developing this ability to judge length at a glance. When children have learned to find the missing rod, the game may be made more challenging by placing two rods on top of a third and finding the rod which fits in the space, or by placing a single rod in the center and finding the one which goes on either side.



Children who have learned to make such judgments by eye seem to know, without hesitation, which rod is needed, unlike adults who usually have to do some figuring. Errors are quickly corrected and children are

free to focus on the relationships of color as well as length. Their awareness of these relationships will be developed if they are asked to *predict* which rod is needed before actually reaching for it:

> Place a brown and a crimson side by side. Which rod is needed with the crimson to make the length of the brown?
>
> If you put two light green rods end to end, which rod will you need to make the same length?
>
> If you have a dark green and place a yellow on top of it, which rod will you need? etc.

Part 2 of Dr. Gattegno's Book I, called Qualitative Arithmetic, contains many suggestions for activities which will help children become more aware of the relationships which exist between the rods.

Fractional Relationships

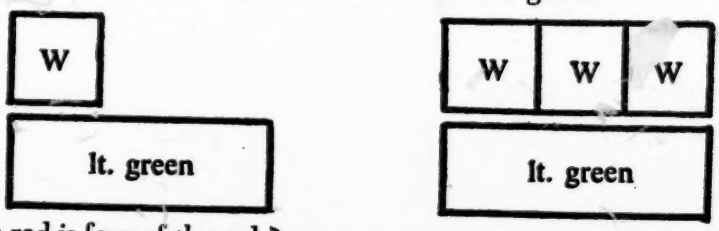
It will be noted that so far we have not called the white 'one', the red 'two', the green 'three', etc. There are good reasons for postponing this usage until later. Any single rod, by itself, is one piece of wood. It can have other numerical values only by comparison with other rods. The crimson, for example, is two of the reds, one half of the brown, four fifths of the yellow, four thirds of the light green and four of the whites. Although its relationship to the white will be the one which is used most frequently, children can see right from the start that this is not the only one there is. When we do not use any fixed numerical values for the rods in this beginning work, children are helped to see that any number is a relationship between two quantities. After they have become familiar with the rods, have learned to estimate length, can build the stair, and name the colors both forwards and backwards, they are ready to answer some of the following questions. These are asked in such a way that the answer can be given by holding up a rod so that every child can answer at the same time.

> Which rod is as long as two of the reds? Put two red rods end to end. Find the rod which is as long as two of the red rods placed end to end.
>
> Which rod is two of the crimsons? Put the two crimson rods end to end. Can you find a rod which is as long as two crimsons?
>
> Which rod is half of the brown? It takes two crimsons to make the length of the brown so we say that the crimson is half of the brown.
>
> Which rod is two of the yellows?
>
> Which rod is half of the orange? etc.

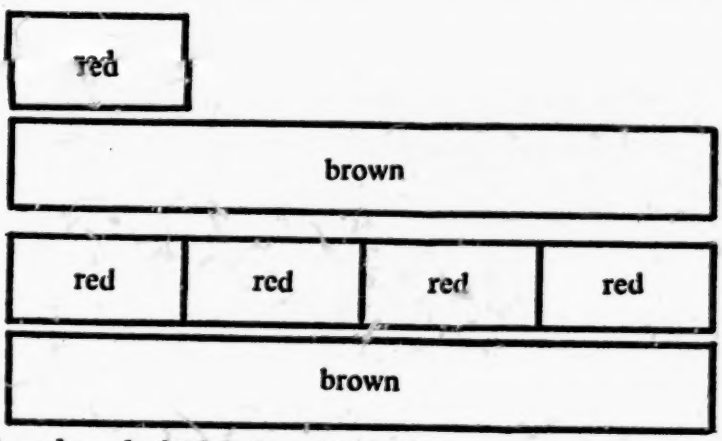
The idea of a half is usually familiar to first graders. The other word they know is quarter. Because of the inconsistency in our language it is not clear that one half means one part out of two, that one quarter means one part out of four, etc. Logically our names for these should be 'one twoth' and 'one threeth', the rest being regular. This irregularity probably prevents children from learning and using these relationships much earlier

than they do. Many children seem quite grateful to have this inconsistency pointed out. Once they appreciate the regularity which is here they can extend their awareness to other relationships quite readily:

Which rod is three of the whites? The green is three of the whites. The white is what of the green? We say that the white is one third of the green because it takes three whites to make a green.



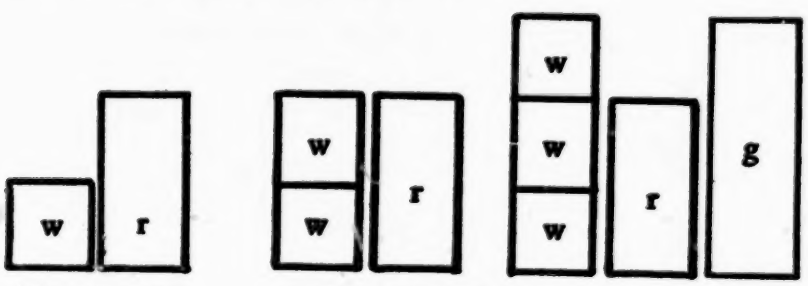
Which rod is four of the reds?
The red is what of the brown? We say that the red is one fourth of the brown because it takes four reds end to end to make the length of the brown.



Hold up the rod which is five of the reds.
The red is what of the orange? etc.

Hold up the rod which is one half of the red. Where is the other half?

Now can you find a rod which is as long as two halves of the red? Which rod is as long as three halves of the red?



The white is half of the red.

Two halves of the red are as long as the red itself.

Three halves of the red are as long as the light green, etc.

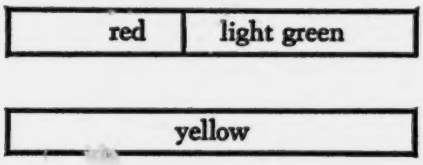
Which rod is four halves of the red, five halves? etc.

With a little more experience children will be able to compare any of the rods with any of the others:

The crimson is what of the yellow?
The black is what of the blue?
The light green is what of the red?
The orange is what of the red? etc.

Addition and Subtraction

If we place a red and a light green rod end to end we will find that a length which is the same as that of the yellow will be formed.



We can say that the red rod plus the green rod equals the yellow rod and can write, $r + g = y$. The words and the symbols used follow the pattern of action where 'plus' (+) refers to the act of putting the two rods end to end and 'equals' (=) means finding one the same length as these two which could be substituted for it. We recommend using the proper mathematical terms, 'plus', 'minus' and 'equals' instead of the expressions, 'and', 'take away', and 'is' which are sometimes used, because these words are a source of confusion for many children, and it is not difficult, with the help of the rods, to learn the meaning of the correct mathematical terms. It should be noted that we are merely applying words and symbols to what the child actually sees, and since there is a direct correspondence between what is done, what is said, and what is written, it becomes clear that a mathematical statement can be made in any of these ways.

A child can understand that many of the things he does with the rods can be looked at from more than one point of view. $r + g = y$ is but one way of looking at these three rods. A person standing on the other side of them would see, reading from left to right still, $g + r = y$. This simple shift in point of view enables children to understand the basic generalization which mathematicians call the law of commutation. While it is not very helpful to try to tell a child in so many words that the sum is independent of the arrangement of its parts, he can see that this is so from what is done with the rods, and the increased awareness he gains from writing these relationships will be useful to him in other situations.

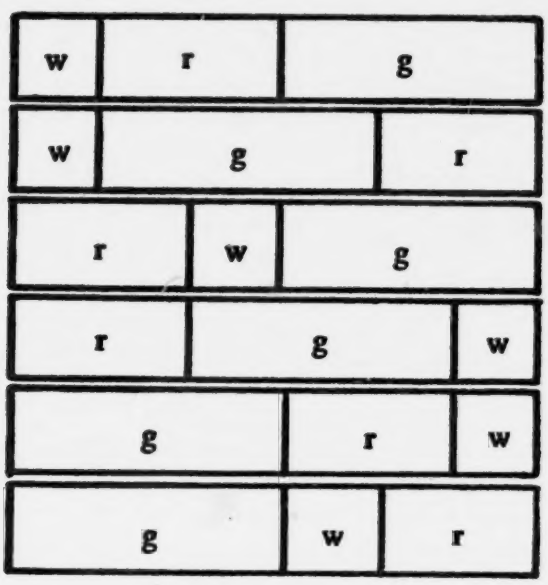
Another person might look at the yellow rod first, however, and say that the yellow rod is equal to the red rod and the green rod placed end to end, or that the yellow is equal to the green and red placed end to end, and these relationships can be written as the others.

If we cover up a portion of the yellow rod by placing a green on top of it, we can find the rod which is needed to complete the length of the yellow, but in this case we will say that the yellow rod *minus* the green rod equals the red rod, and can express the same thing in symbols as before. The child can see that what is meant is the *difference* between two quantities. This meaning of subtraction is more general and therefore more useful than the concept of 'take away' with which so many children have difficulty. The following relationships, therefore, are all derived from this simple arrangement of three rods:

$r+g=y$	$y=g+r$	$y-g=r$	$r=y-g$
$g+r=y$	$y=r+g$	$y-r=g$	$g=y-r$

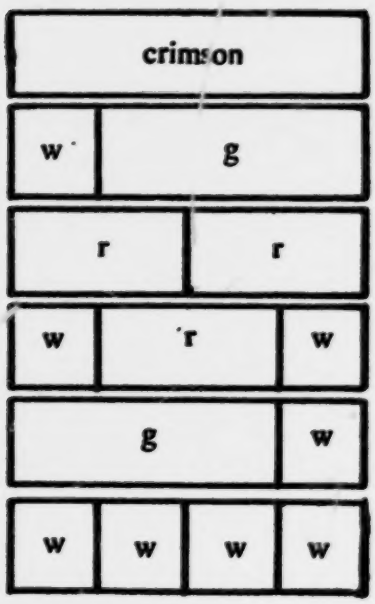
When the child understands that these general relationships will be the same no matter which numbers are used he will not need to waste time learning the specific combinations of arithmetic in unrelated form. At a later stage, perhaps in the third grade, we can invent some imaginary rods, say apricot, banana, and charcoal colored ones, so that the length of the first two placed end to end will form the third. We can then make the following statement, $a + b = c$, which immediately can be expressed as, $b + a = c$, $c - b = a$, $2a + 2b = 2c$, etc. Children who have had sufficient experience with the rods are ready to make such statements, expressing what they have learned in algebraic form. Algebra becomes a way of summarizing what is already familiar rather than a separate subject whose study must be postponed because it is too abstract.

In making and recording their number patterns children will discover that different arrangements of the same rods will produce the same results. If, for example, we look for all the possible arrangements of the white, the red, and the light green we have:



Six year olds enjoy making these different permutations. Some children will recognize that only six are possible with three rods and, either at this stage or later on, will be able to figure out how many different permutations they would have for four, five, or six rods. They should be encouraged to look carefully at what they have done and to see if they can discover the general principle which is involved.

After children have learned to estimate length at a glance they are ready to make the pattern of each rod and to 'read' what they see:



c = c, or c + 0 = c

w + g = c

r + r = c, or 2 × r = c

w + r + w = c

g + w = c

w + w + w + w = c,
or 4 × w = c

This is an activity in which the whole class can share profitably, for while one child is reading his pattern the other children can check by looking at their own or by closing their eyes and seeing if the combinations given are correct. A slightly different approach would be for the teacher to read one of the patterns, not without error, and have the children tell whether a given statement is correct or not.

In order to use the rods for processes with whole numbers it is necessary to use one unit of measure consistently. Children can quickly learn to make comparisons with the white, operating from what they already know about the relationships between the rods:

The red is what of the white?
The crimson is what of the white?
The brown is what of the white?
The yellow is what of the white?
The orange is what of the white?
The light green is what of the white? etc.

After a short experience with this we can just explain that we are going to leave out 'of the white' unless we mean something else.

Now we can write the pattern of each rod in a new way. Since the way in which these are written corresponds directly with what the child has done with the rods, it is largely a question of transcribing something which is already familiar. If we re-write the relationships of the red, green, and yellow, which we have already considered in a more general way (see pages 7 and 8), we have the following statements:

$3+2=5$ $\quad$ $5-3=2$ $\quad$ $\begin{array}{r} 3 \\ +2 \\ \hline 5 \end{array}$ $\quad$ $\begin{array}{r} 2 \\ +3 \\ \hline 5 \end{array}$ $\quad$ $\begin{array}{r} 5 \\ -3 \\ \hline 2 \end{array}$ $\quad$ $\begin{array}{r} 5 \\ -2 \\ \hline 3 \end{array}$

$2+3=5$ $\quad$ $5-2=3$

$5=2+3$ $\quad$ $2=5-3$

$5=3+2$ $\quad$ $3=5-2$

Furthermore, these statements can be completed no matter where the unknown appears: $\square+3=5$ or $2+\square=5$, as well as the more familiar $3+2=\square$. Or we may express the same relationship as, $\frac{3}{2}\times 2+\frac{2}{3}\times 3=5$. Again, these statements are just different ways of looking at the same thing, and it is extremely wasteful to try to remember them separately. A child who is encouraged to form so-called 'number-bonds' by memorizing combinations in unrelated form has a task which becomes increasingly difficult. By calling for greater flexibility from the very first we make it difficult for a child to use such a blind approach effectively. This is not to say that the 'basic combinations' should not be remembered, but rather that what one remembers should be useful in a wide range of situations. Instead of repeating the same problems again and again, therefore, these relationships can be presented in a variety of combinations so that there is no drill in the usual sense. Dr. Gattegno's Book I gives an indication of how computational skill can be developed without the usual repetition.

Sometimes children whose mathematical ideas are very clear have trouble, at first, in reading mathematical statements. Their problem is similar to that of learning to read English, though less complicated. When they learn to read their own patterns from the rods, transcribe them into numerals, and then read them back, their troubles will be over. Because of the flexibility which is used in writing equations it is impossible for a child to go very far without understanding what he is doing.

Children soon realize that a given number can be called by a variety of names and enjoy making up new ones of their own. Here are some of the names which we can give to four:

$2+2$	2×2	$10-6$	$\frac{2}{3}\times 6$	
$\frac{1}{2}\times 8$	$3+1$	$33-29$	$\frac{2}{5}\times 10$	
$\frac{4}{5}\times 5$	$\frac{4}{3}\times 3$	$\frac{4}{9}\times 9$	$\frac{4}{7}\times 7$	etc.

At a slightly later stage children will be able to figure out the operator which will be needed to produce four from any given number: $\square \times 10 = 4$, etc.

It will be noted that we have worded our statements or questions in such a way that they can be transcribed directly into an equation. There is one kind of statement, however, which is ambiguous and should be avoided. We would prefer to say, 'The yellow is what of the black?', which can be written directly as, $y = \square \times b$, rather than saying, 'What is the yellow of the black?', which one could interpret as, $\square = y \times b$, and is not the meaning intended. This first form of the question works equally well if we wish to consider the reciprocal relationship and ask, 'The black is what of the yellow?', for this becomes, $b = \boxed{\frac{7}{5}} \times y$. Such expressions as, five *out* of seven ($\frac{5}{7}$), or seven *as measured by* five ($\frac{7}{5}$), are also helpful in making clear what is meant. While there is some ambiguity, in the early grades, in reading '×' as 'of' when preceded by a fraction, and as 'times' in the case of a whole number, it is not easily avoided.

Multiplication

We have seen how the rods can be particularly useful in the child's early experiences with fractions. Multiplication, also, can be started much earlier and in quite a different way with the rods. As soon as a child starts to write the pattern of any given rod, he is ready to use the language and notation of multiplication to describe what he sees. If for one of his lines of his six pattern, for example, he should read, 'two plus two plus two', we would write it, $2+2+2$, but if he should say that he sees three twos we would use the notation, 3×2, reading it 'three times two', or simply, 'three twos'. Some children will learn a great deal from this kind of exposure, and others will soon learn by using the rods the distinction between 2×4 and 4×2, 3×5 and 5×3, etc.

Multiplication follows naturally from grouping rods of the same color together, whether in a line or rectangle. For any given rectangle we can ask, 'How many whites will it take to cover this?' At a later stage the same problem can be posed in terms of area or volume.

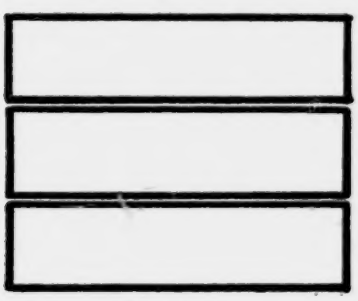
> How many different rectangles can you make, using rods all of the same color which would be the same as 15 whites?
>
> Place one of the white cubes on the back of your hand. The part that is touching your skin is called one square centimeter. How many different rectangles can you make having an area of 15 square centimeters, using rods of a single color?

The amount of wood in the white cube is called one cubic centimeter. How many different rectangles can you make, using rods all of the same color, which will have 15 cubic centimeters of wood?

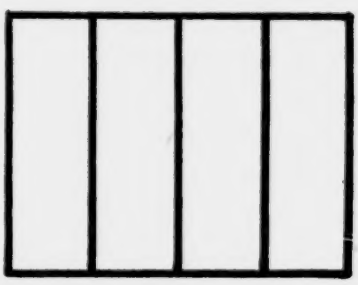
Using three of one color and three of another, make a rectangle having 15 cubic centimeters of wood.

Using five of one color and five of another, make a rectangle having 15 cubic centimeters of wood.

If we make a rectangle with the crimsons to represent 3×4, we can show that a rectangle made of the greens, 4×3, gives the same result by placing one on top of the other.



3×4



4×3

Here again we are meeting the law of commutation. After the children have seen it in a number of applications they may be ready to generalize it for themselves. A somewhat sceptical attitude on the part of the teacher can be more helpful than a readiness to put into words what students are beginning to discover for themselves. They should be encouraged to make as many tests of the idea as they need in order to feel sure that it always works. Here again we have two different ways of looking at the same thing. The children can see that this is so if a group of dots are placed on a piece of tagboard and rotated through 90 degrees:

3×5

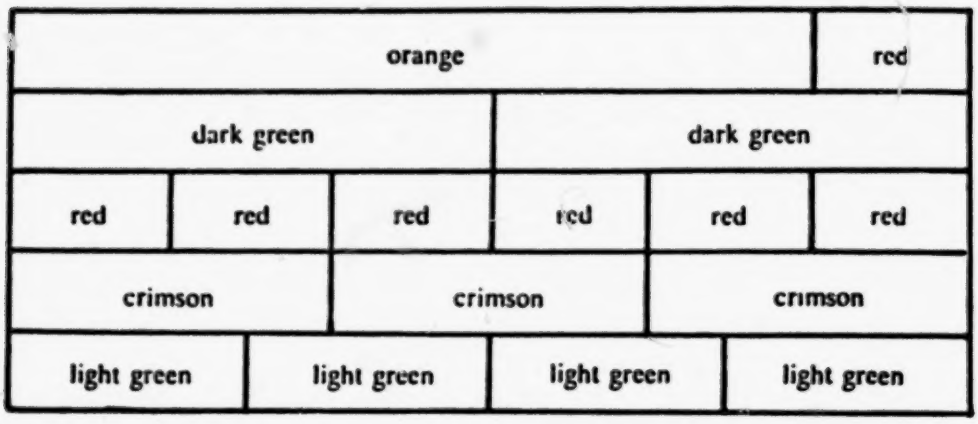
5×3

If we focus first on the number of horizontal rows and then on the amount in each row we have 3×5 in the first case and 5×3 in the second, though this rotation has not changed the quantity.

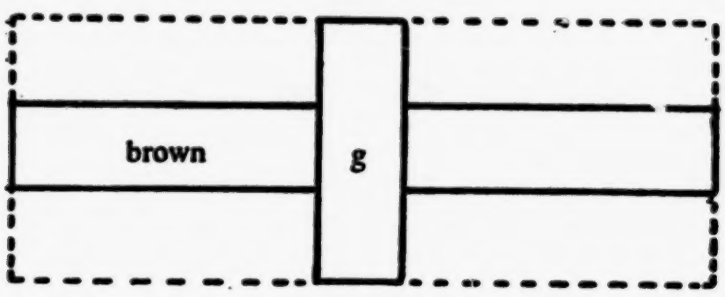
The Study of Factors

The learning of multiplication proceeds through a study of products and their factors rather than the memory of 'multiplication tables'. Using the rods one can determine what the factors of a given number are by

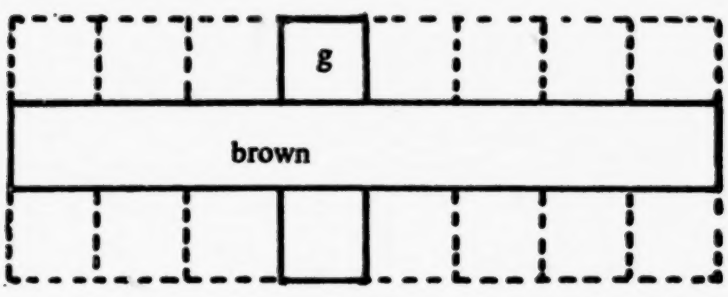
seeing if the length can be made by a line of rods having all the same color. If a given length can be formed by the whites and no other, then the number is a prime number.



The factors of twelve, for example, are six, two, four and three. Since one is a factor of every whole number it need not be included. Children enjoy discovering for themselves which numbers are prime and which are not. With the help of 'towers' which enable the simultaneous study of multiplication, division, and the fractional parts of products, children make rapid progress. These multiplication towers can be introduced as a symbol for a rectangle made up of the same color of rod. Instead of placing eight greens side by side to represent 8×3, for example, we can make a tower by crossing the three with the eight. 3×8, on the other hand will be represented by placing a three across the eight:



3×8



8×3

The rod on the top indicates how many of the other rods there are.

Children will probably need to construct a number of those rectangles before they understand the symbolism which is being used. Once they do they will be able to read off a number of relationships from any given tower. From the two above we have:

$3 \times 8 = 24$ $\quad$ $24 \div 8 = 3$

$8 \times 3 = 24$ $\quad$ $24 \div 3 = 8$

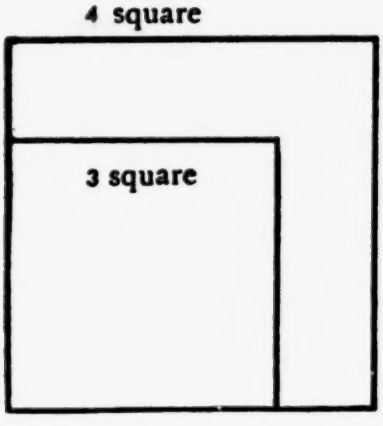
$\frac{1}{8} \times 24$

$\frac{1}{3} \times 24$ $\quad$ $\frac{2}{3} \times 24$ $\quad$ $\frac{3}{3} \times 24$

$\frac{3}{8} \times 24$ $\quad$ $\frac{7}{8} \times 24$

$\frac{3}{4} \times 24$ $\quad$ etc.

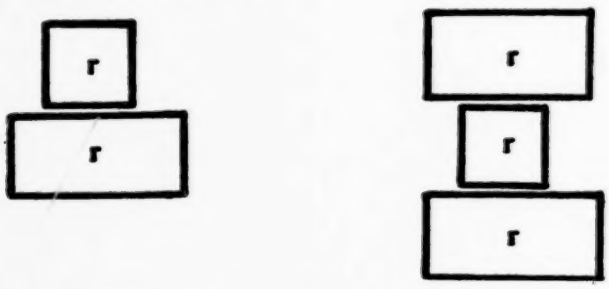
Children in the early grades will enjoy constructing squares and cubes, using rods of the same color. The cubes can be held together with rubber bands so that they may be examined more easily. Having discovered for themselves which numbers are 'square numbers', some of the children may be interested in exploring the difference between successive squares. This relationship can be shown very nicely by using a meter stick with rods to mark the squares, a red standing for 2^2 at 4, a green standing for 3^2 at 9, etc. When the next smaller square is placed on top of any given square, the structural relationship between the two becomes clear:



$4^2 - 3^2 = 2 \times 3 + 1$

$16 - 9 = 7$

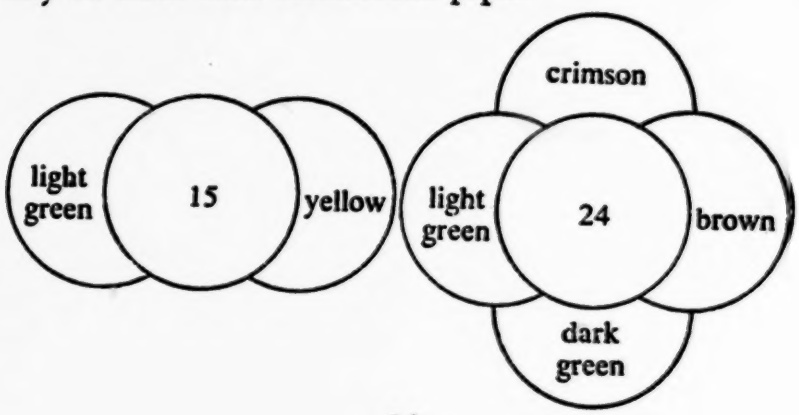
Four may be represented by a tower of two reds (2×2) and eight by a tower of three reds ($2 \times 2 \times 2$):



2×2 or 2^2 (Front view) $2 \times 2 \times 2$ or 2^3

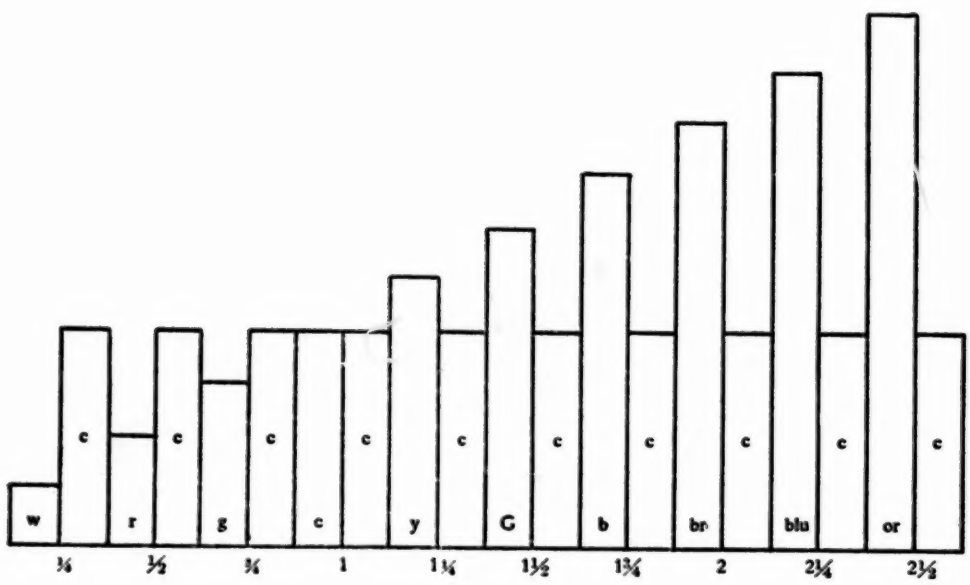
When children realize that the four in any tower can be replaced by two twos and the eight by three twos, they can find the answer to any multiplication problem involving four or eight by doubling. Four times eight becomes $2 \times 2 \times 8$ and eight times eight can be found by starting from eight and doubling three times, $2 \times 2 \times 2 \times 8$. Some may, at first, have trouble doubling sixteen in their heads. They should be encouraged to double the ten first and then the six, adding the twenty and twelve. For some reason doubling seems to have quite a fascination for children and they soon become quite skillful at it. It is an interesting exercise to start with any number from one to ten, see how far one can go by doubling and then return to the starting point by halving. As children become familiar with the towers they will realize that any number can be written as the product of prime numbers. In the towers 8×9, for example, we can replace the 8 with $2 \times 2 \times 2$ or 2^3 and the 9 by 3×3 or 3^2. 72 is therefore $2^3 \times 3^2$.

A child's familiarity with products and their factors will be increased by studying fractional parts and division simultaneously. His ability to compute rapidly can be developed by using cards on which the products are represented by the colors of their factors. Each factor of a given number is indicated opposite the other one. Since the cards can be rotated the order is of no significance. Products which have two sets of factors are shown with four colors. These cards can be obtained from the Cuisenaire Co. or may be made from construction paper.



It will be recognized that the cards are a further abstraction of the towers, so that the rods may be used, if necessary, to figure out the products which are represented. When two or more children sit down with a pack of these cards and play the game of seeing who can call out the product first, progress will be rapid. Since a child knows many different ways of looking at the same number, his experience is considerably richer than it would be if he had memorized tables. He is also able to figure out any product which he does not remember.

When we measure any length, we tell how many, or what part, of the unit is equal to the length we are measuring. If there is a part left over it is expressed as a fraction of the unit. We can choose any one of the rods as our unit of measurement and compare all the other rods in the stair to it. Using the crimson rod as our unit we have:

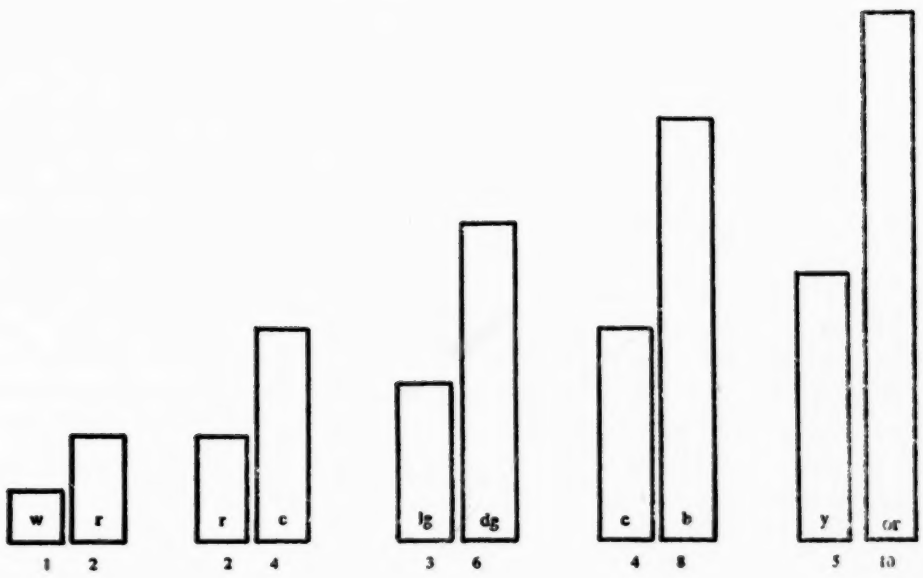


It is very helpful for children to see this whole array of rods in this way, for it helps them to develop a more general structure of relationships without becoming lost in particulars. They will see that $\frac{1}{2}$ and $\frac{2}{4}$ are names for the same relationship and that $\frac{5}{4}$ and $1\frac{1}{4}$ are also the same, without their having to develop rules for going from one to another. The ideas involved in measuring all the rods in the staircase by one which is chosen as the unit are ideas which most students find difficult when they are presented by traditional methods. Again, these relationships which are seen by the child can be expressed in words or in writing. Second grade is not too early for children to write:

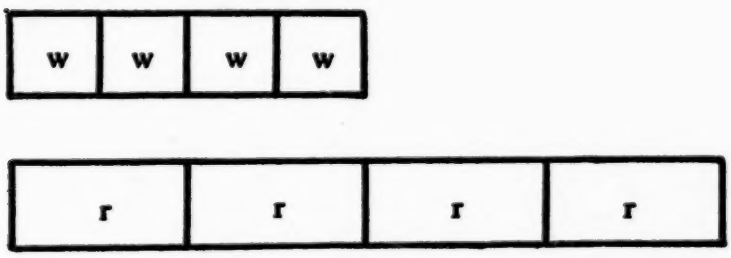
$c = \frac{1}{2} \times br$ $\quad r = \frac{1}{3} \times G$ $\quad c = \frac{4}{5} \times y$

$br = 2 \times c$ $\quad G = 3 \times r$ $\quad c = \frac{3}{4} \times g$ $\quad$ etc.

If we use the rods to show the relationship $\frac{1}{2}$, or (1, 2) written as an ordered pair we have the following array:



One half is the relationship of one out of two, two out of four, three out of six, etc. Other pairs, which are equivalent to the first, can be generated by increasing each term in the same proportion:



$$\frac{1+1+1+1}{2+2+2+2} \quad \text{or} \quad \frac{4\times 1}{4\times 2}=\frac{4}{8}$$

$$\frac{1+1+1+1}{2+2+2+2} \quad \text{or} \quad \frac{4\times 1}{4\times 2}=\frac{4}{8}$$

A more abstract approach to operations with fractions, suitable for children who have had considerable experience with the rods, is given in Dr. Gattegno's Book VI.

Because children are unencumbered by certain ways of looking at things which result from the training in arithmetic which most adults have had, there may be times when they will be learning more rapidly than their teachers and will see relationships which are not apparent to the more experienced person. If the teacher is not threatened by such a situation, can acknowledge her own ignorance, and explore the situation along with her students, a great deal can be gained by all. In any case it is important for the child to realize that the question of what is correct or not correct is decided by the nature of reality, which in most cases can be checked by an experiment with the rods, rather than by an arbitrary ruling from the teacher. This greater measure of independence which children have, this freedom, is undoubtedly one of the reasons that so much more can be accomplished than we have previously attempted.

People are often surprised to find that we are ready to introduce some topics much earlier than we have in the past, and point out that most schools have trouble accomplishing what they set out to do as is. Such a program is possible under certain conditions and it has therefore been necessary to talk about a child's approach to learning as well as about what can be done with the rods. The rods, after all, possess no intelligence of their own, so that the results which one obtains with them must depend on how they are used. We are particularly interested in the kinds of learning out of which the child gets the most 'mileage', for we know that it is possible for mathematics to become easier and easier for a child as he goes along in school, instead of more and more difficult as is so often the case at present. The flexibility which is possible with the rods encourages the child to look at the same problem from more than one point of view and tends to make unprofitable an approach which relies primarily on memory. For this reason the teacher should expect to find that the children who try to depend on memory and a fixed way of doing things will have difficulties. The healthiest atmosphere is one in which such difficulties can be acknowledged without anxiety, so that the child is able to recognize that he has not learned, but that his teacher expects him to be able to. Learning proceeds so rapidly when the conditions are just right that it can be quite wasteful to hold back a whole class because a few children seem to be having difficulty. First graders, particularly, can be quite unpredictable at this stage of their learning, and children who were having trouble one week may move rapidly ahead the next. A relaxed and accepting atmosphere, which at the same time helps a child recognize that he is responsible for his own learning, can avoid the necessity of having a great deal of repetition or individual help. Since a child can check his own work with the rods, as soon as he learns to read equations he is ready for some independent work. This self-checking feature makes it much easier for a teacher to handle individual differences

than it is in situations where the child is directly dependent upon the teacher to tell him when he is right or wrong.

Since this paper is intended as an introduction only, a number of topics have been indicated to show some of the possibilities with the rods without trying to give a detailed treatment.

A limited number of extra copies of this article may be obtained from the National Council of Independent Schools, 84 State Street, Boston 9, Mass.

The rods and Dr. Gattegno's series of books may be obtained from the Cuisenaire Company of America, 246 E. 46th St., New York 17, N. Y.